Growth Theory Handout

The level of output per worker in an economy on its steady-state growth path is:

$$Y_t / L_t = \kappa^{*\lambda} \times E_t = \left(\frac{s}{n+g+\delta}\right)^{\lambda} \times E_t$$

Where:

κ^* is the steady-state capital-output ratio equals $s/(n+g+\delta)$

κ is K/Y, the capital-output ratio

λ, the growth multiplier, equals $\alpha/(1-\alpha)$

α is the diminishing-returns-to-scale parameter in the production function $Y/L = (K/L)^{\alpha} \times E^{1-\alpha}$

s is the economy's savings rate

n is the economy's rate of labor force growth

δ is the rate at which the capital stock depreciates.

If the economy is not on its steady-state growth path, it heads toward its steady-state growth path at a proportional rate of $(1-\alpha) \times (n+g+\delta)$, closing that proportion of the gap between its current capital-output ratio and the steady-state capital-output ratio each year.

Deriving the (continuous-time) law of motion for the capital-output ratio κ_t.

At any instant, the rate of change of the capital stock K_t is equal to the rate at which new investment is adding to the capital stock minus the rate at which depreciation is subtracting from the capital stock:

$$\frac{dK_t}{dt} = s \times Y_t - \delta \times K_t$$

Thus the proportional rate of change of the capital stock is:

$$\frac{1}{K_t}\frac{dK_t}{dt} = s \times \frac{Y_t}{K_t} - \delta$$

From the production function:

$$Y_t = K_t^{\alpha} \times L_t^{1-\alpha} \times E_t^{1-\alpha}$$

We can calculate the proportional rate of change of output:

$$\frac{dY_t}{dt} = \alpha\left(\frac{Y_t}{K_t}\right)\frac{dK_t}{dt} + (1-\alpha)\left(\frac{Y_t}{L_t}\right)\frac{dL_t}{dt} + (1-\alpha)\left(\frac{Y_t}{E_t}\right)\frac{dE_t}{dt}$$

$$\frac{1}{Y_t}\frac{dY_t}{dt} = \alpha\left(\frac{1}{K_t}\frac{dK_t}{dt}\right) + (1-\alpha)(n+g)$$

From these, we can derive the proportional rate of change of the capital-output ratio κ_t by simple subtraction:

$$\left(\frac{1}{\kappa_t}\frac{d\kappa_t}{dt}\right) = \left(\frac{1}{K_t}\frac{dK_t}{dt}\right) - \left(\frac{1}{Y_t}\frac{dY_t}{dt}\right)$$

$$\left(\frac{1}{\kappa_t}\frac{d\kappa_t}{dt}\right) = \left(\frac{1}{K_t}\frac{dK_t}{dt}\right) - \alpha\left(\frac{1}{K_t}\frac{dK_t}{dt}\right) - (1-\alpha)(n+g)$$

$$\left(\frac{1}{\kappa_t}\frac{d\kappa_t}{dt}\right) = (1-\alpha)\left[\left(\frac{1}{K_t}\frac{dK_t}{dt}\right) - (n+g)\right]$$

$$\left(\frac{1}{\kappa_t}\frac{d\kappa_t}{dt}\right) = (1-\alpha)\left[\frac{s}{\kappa_t} - \delta - (n+g)\right]$$

And then multiplying to get the rate-of-change of the capital-output ratio by itself on the left hand side:

$$\frac{d\kappa_t}{dt} = (1-\alpha)\left[s - (n+g+\delta)\kappa_t\right]$$

Let's call s/(n+g+δ) by the name κ*:

$$\frac{d\kappa_t}{dt} = -(1-\alpha)\left[(n+g+\delta)(\kappa_t - \kappa^*)\right]$$

This kind of differential equation has a particular solution, where κ_0 is the value of κ when t=0 and e is 2.71828…:

$$\kappa_t = \kappa^* + (\kappa_0 - \kappa^*) \times e^{-(1-\alpha)(n+g+\delta)t}$$

Memo:

$$e^0 = 1.000$$
$$e^{-.1} = 0.905$$
$$e^{-.7} = 0.497$$
$$e^{-1} = 0.368$$
$$e^{-2} = 0.135$$