Economics 202a, Fall 1998
Lecture 3: Solow Model II Handout

Basics:

$$Y_t = F(K_t, A_t L_t)$$

$$\frac{dA_t}{dt} = gA_t$$

$$\frac{dL_t}{dt} = nL_t$$

$$\frac{dK_t}{dt} = sY_t - \delta K_t$$

The model solution, for the general production function $y_t = f(k_t)$:

$$\frac{dk_t}{dt} = sf(k_t) - (n + g + \delta)k_t$$

$$y^* = f(k^*)$$

$$Y^*_t = A_t L_t y^*$$

$$K^*_t = A_t L_t k^*$$

$$\frac{K^*_t}{Y^*_t} = z^* = \frac{k^*}{y^*}$$

$$\frac{Y^*_t}{L_t} = A_t y^*$$

$$\frac{K^*_t}{L_t} = A_t k^*$$

The model solution, for the specific production function $y_t = k_t^\alpha$:

$$\frac{K_t}{Y_t} = \frac{k_t}{y_t} = z_t = \frac{s}{n + g + \delta} + \left[\frac{K_0}{Y_0} - \frac{s}{n + g + \delta}\right] e^{(\alpha - 1)(n + g + \delta)t}$$

$$Y_t = A_t L_t (z_t)^{\frac{\alpha}{1-\alpha}}$$

$$\frac{Y_t}{L_t} = A_t (z_t)^{\frac{\alpha}{1-\alpha}}$$

$$y_t = (z_t)^{\frac{\alpha}{1-\alpha}}$$

There are still other things that we want to look at: for example, the real wage. If we assume that factors of production are paid their marginal products, then the real wage at any time t is:

$$w_t = \frac{d(F(K_t, A_t L_t))}{dL} = A_t f(k_t) - \frac{K_t}{L_t} f'(k_t)$$

$$r_t = \frac{d(F(K_t, A_t L_t))}{dK} = f'(k_t)$$

$$w_t L_t + r_t K_t = A_t L_t f(k_t) = Y_t$$

As it should from constant returns to scale. An alternative way to write the real wage is:

$$w_t = A_t[f(k_t) - k_t f'(k_t)]$$

$$r_t = f'(k_t)$$

which makes it more clear that the real wage is growing at rate g, and the real return to capital is constant in steady-state growth.

If we restrict our attention to Cobb-Douglas production functions, then:

$$w_t = A_t[f(k_t) - k_t f'(k_t)] = A_t[k_t^{\alpha} - k_t \alpha k_t^{\alpha-1}]$$

$$w_t = (1-\alpha)A_t k_t^{\alpha} = (1-\alpha)A_t z_t^{\frac{\alpha}{1-\alpha}}$$

$$r_t = f'(k_t) = \alpha k_t^{\alpha-1} = \alpha z_t^{-1}$$

In other words, a share α of output is going to capital--and is evenly split as the return to capital. A share $1-\alpha$ of output is going to labor--and is evenly split as well.

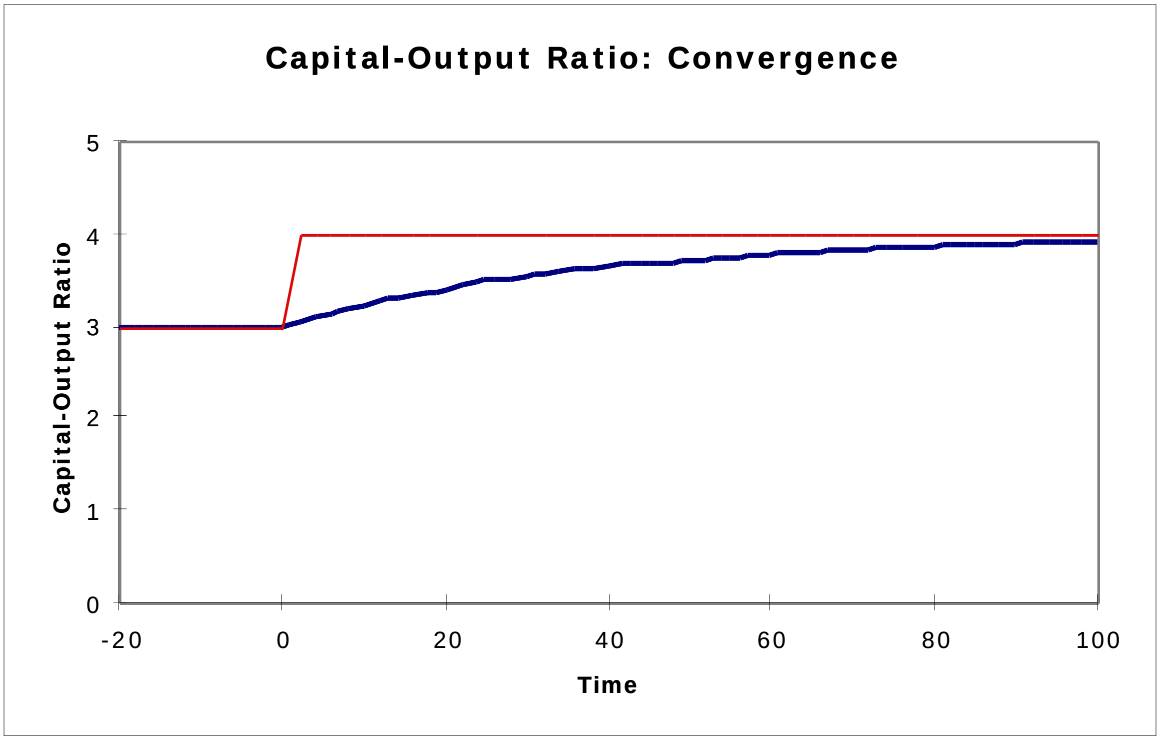
And now let's look at what happens when we change the parameters of the model:

I. A change in the savings rate s:

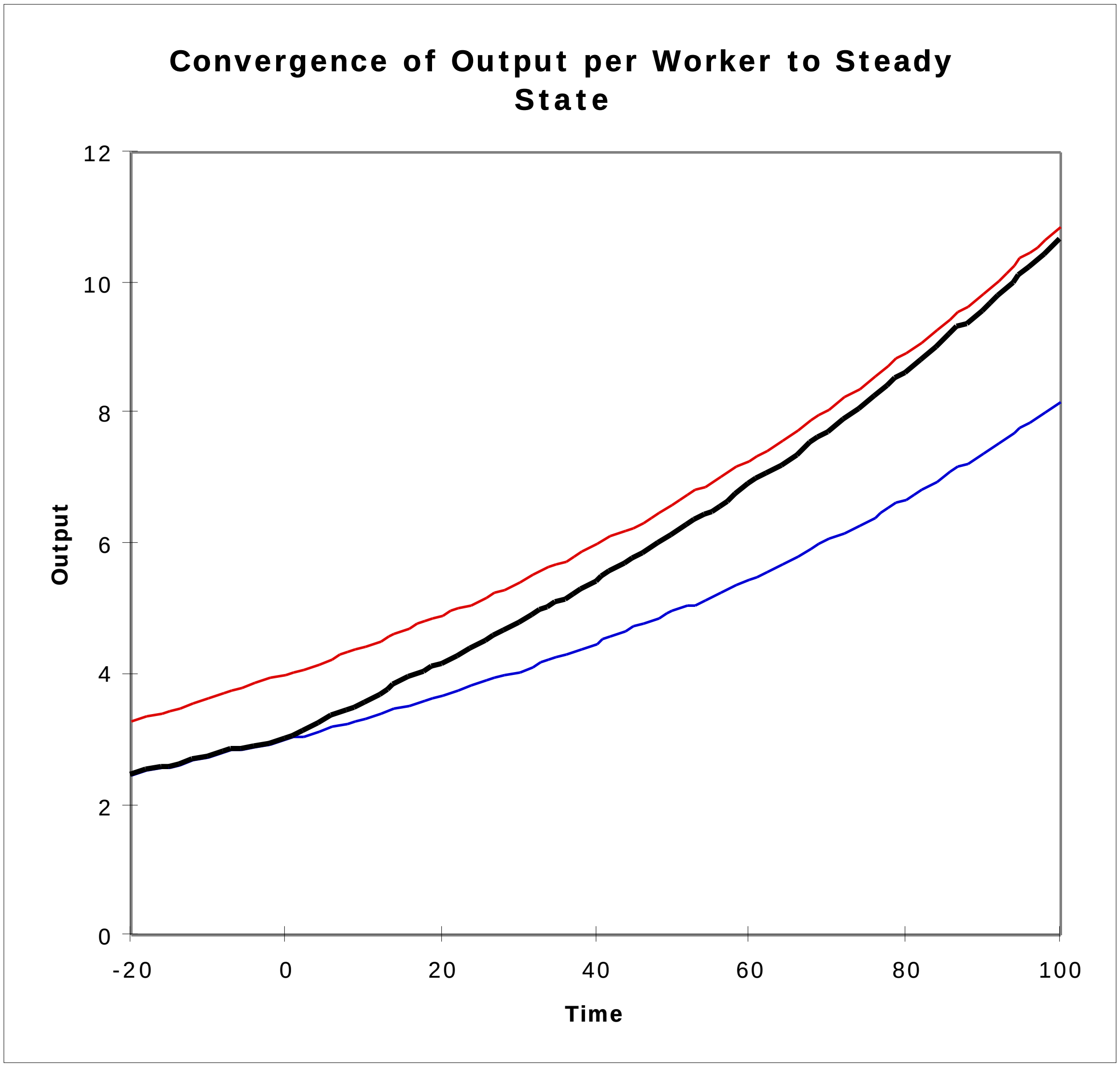
Let's take α = .5; n=g=.01; δ=.03; s starts at .15 and jumps to .20.
Start at time zero with A_0=L_0=1…

Our steady-state growth path then jumps from z* = 3, with Y_t growing at .02 per year, to z* = 4, with Y_t growing at .02 percent per year--a 33.3% boost to steady-state capital per effective worker, and a 33.3% boost to output per effective worker.

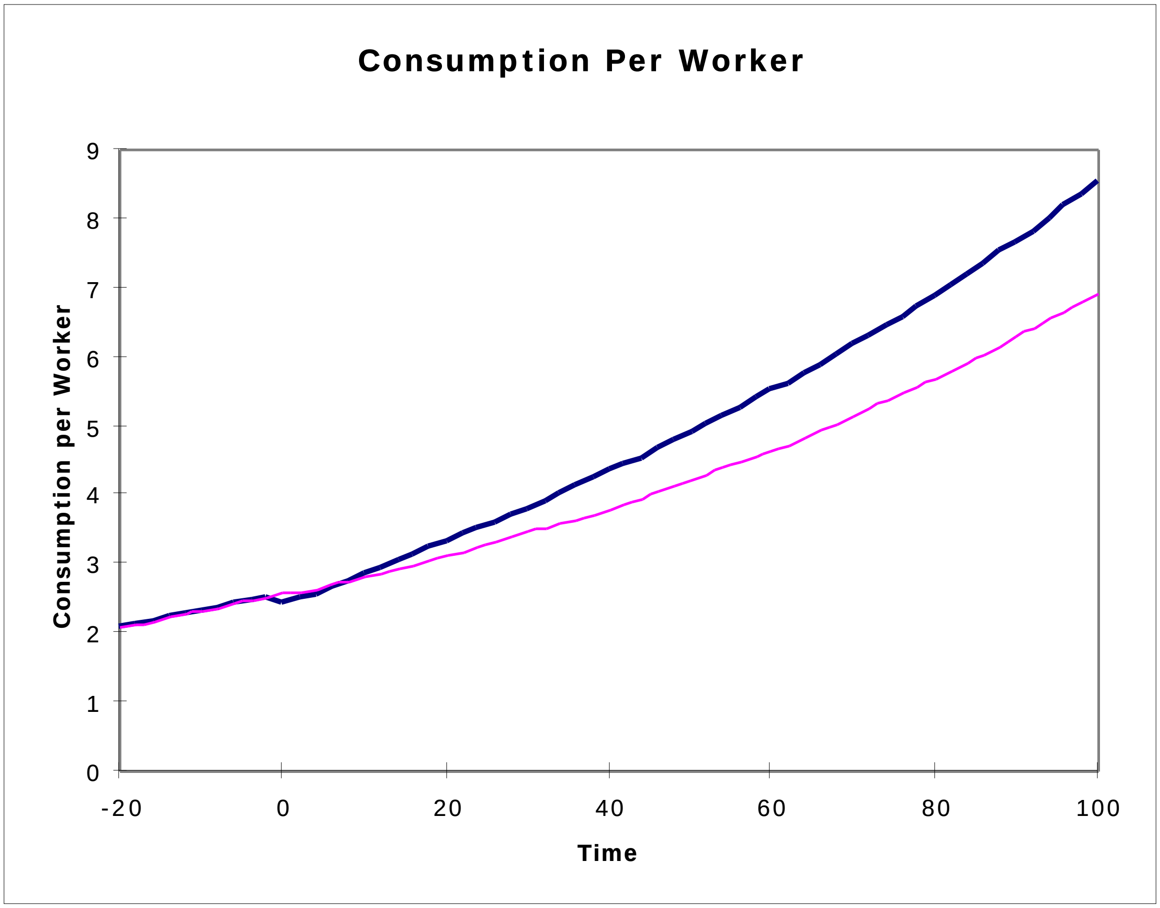
How fast does this economy converge to its steady state? (1- α)(n+g+δ) = .025 equals 2.5 percent per year. So the 1/e time--the time after the time 0 jump in the savings rate for the capital-output ratio to close the gap to its new steady-state value to 1/e of its initial value--is 40 years.



Capital-Output Ratio: Convergence
Capital-Output Ratio
5
4
3
2
1
0
-20
0
20
40
60
80
100
Time



Convergence of Output per Worker to Steady State
Output
Time
0
2
4
6
8
10
12
-20
0
20
40
60
80
100



Consumption Per Worker
Consumption per Worker
9
8
7
6
5
4
3
2
1
0
-20
0
20
40
60
80
100
Time

"Level effect" but not a "growth effect"...

Does consumption rise when the savings rate rises? Answer: at the beginning, certainly not. Eventually--it depends.

Consumption per effective worker along a steady-state growth path:

$c^* = y^*(1-s) = y^* - (n+g+\delta)k^*$

What happens if we differentiate $c^*(s, n, g, \delta)$ as a function of the model parameters with respect to the savings rate s?

$$\frac{dc^*}{ds} = \frac{d}{ds} f(k^*) - \frac{d}{ds}(n+g+\delta)k^*$$

$$\frac{dc^*}{ds} = f'(k^*)\frac{dk^*}{ds} - (n+g+\delta)\frac{dk^*}{ds}$$

$$\frac{dc^*}{ds} = \left[f'(k^*) - (n+g+\delta)\right]\frac{dk^*}{ds}$$

where, as I said before, the derivatives are taken understanding the steady-state values to be functions of the model parameters.

For the *sign* of the effect of an increase in the savings rate on steady-state consumption, we can ignore the last term. It will be positive. Whether the whole right-hand side is positive depends on comparing $f'(k^*)$ to $(n+g+\delta)$--depends on whether the marginal product of capital at the steady state (for that is what $f'(k^*)$ *is*) is greater or less than the sum: $(n+g+\delta)$.

It can go either way. If $f'(k^*) < (n+g+\delta)$, then the economy is called *dynamically inefficient*--consumption could be raised at all future dates by lowering saving today. If $f'(k) = (n+g+\delta)$, the economy is said to be at the *Golden Rule*, which we will talk about more later on.

Can we get any further in evaluating the effect of an increase in savings on steady-state consumption? We can if we assume that the production function is Cobb-Douglas.

In intensive form, then:

$$y^* = k^{*\alpha} = (z^* y^*)^{\alpha}$$

$$y^* y^{*-\alpha} = z^{*\alpha}$$

$$y^* = (z^*)^{\frac{\alpha}{1-\alpha}} = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

And steady-state consumption per effective worker:

$$c^* = (1-s)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

So consumption per worker along the steady-state growth path is:

$$\frac{C_t^*}{L_t} = A_t(1-s)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

And the derivative of c*, taken to be a function of the parameters of the model, with respect to s is:

$$\frac{dc^*}{ds} = -\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} + \left(\frac{\alpha}{1-\alpha}\right)(1-s)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}-1}\left(\frac{1}{n+g+\delta}\right)$$

This will be positive if and only if:

$$-1 + \frac{\alpha}{1-\alpha}\left((1-s)\frac{1}{S}\right) > 0$$

$$s < \alpha$$

This gives you a definite test. And it tells you that:

- When s is sufficiently low, increases in s increase steady-state consumption.

- When s hits the magic number of the capital share α, you are at the *Golden Rule.*
- When s is higher, increases in s decrease steady-state consumption and the economy is *dynamically inefficient.*
- Increases in α increase the level of s at which the economy becomes dynamically inefficient.

II. A change in the population growth rate n:

Let's take α = 1/3; n=g=.01; δ=.03; s = .25.; and suppose suddenly n drops to 0
Start at time zero with $A_0=L_0=1$…

Our steady-state growth path then jumps from z* = 5, with Y_t growing at .02 per year, to z* = 6.25, with Y_t growing at .02 percent per year--a 25% boost to steady-state capital per effective worker, and an 11.8% boost to steady-state output per effective worker.

How fast does this economy converge to its steady state? (1- α)(n+g+δ) = .026667 equals 2.67 percent per year. So the 1/e time--the time after the time 0 jump in the savings rate for the capital-output ratio to close the gap to its new steady-state value to 1/e of its initial value--is 37.5 years.

Speed of convergence II

Let me note equation (1.25) on page 22 of Romer's *Advanced Macroeconomics*: "in the vicinity of the balanced growth path, capital per unit of effective labor converges toward k* at a speed proportional to its distance from k*... [with] $\lambda = (1-\alpha_k)(n+g+\delta)$. Why develop this when we already have our exact solution for z_t? Because our solution for zt is only for the Cobb-Douglas case. What Romer produces is much more general: that if the economy is "near" its steady-state growth path, its convergence behavior is "nearly" the same as Cobb-Douglas.

The Solow Model and the Central Questions of Growth Theory

We ahve seen that only growth in the effectiveness of labor can lead to *permanent* growth in output per worker, and that for reasonable caes the impact of changes in capital per worker on output per worker is modest. Thus only differences in the effectiveness of labor have any reasonable hope--in this framework--of accounting for vast differences in wealth across time and space.

If the returns that capital commands in the markets are a rough guide to its contributions to output, then variations in the accumulation of physical capital do not account for a significant part of either worldwide economic growth or cross-country income differentials.

- differences in savings rates too small to account for cross-section
- differences in rates of return on capital observed in the world are not large enough for capital play a big role: a ten-fold difference in output per worker arising from differences in capital per worker implies a hundredfold difference in the (gross) marginal product of capital.
- the Solow model does not identify what the "effectiveness of labor" is.

Growth accounting:

$$\frac{d}{dt}\ln\left(\frac{Y_t}{L_t}\right) = \alpha_k(t)\left[\frac{d}{dt}\ln\left(\frac{K_t}{L_t}\right)\right] + R(t)$$

Convergence:

if the A's are the same (or become the same), and if s's are similar, than countries should become more alike.

Feldstein-Horioka: does it still hold?

Mankiw, Romer, and Weil--implied α of 0.60 for their economies in cross-section (neglecting reverse causality).

II. Classical and NeoClassical Growth Models

Thursday Jan 22: The Solow Growth Model

Lecture 2 handouts: Solow model and problem set 1

David Romer, Advanced Macroeconomics, pp. 5-15.
Robert Solow (1956), "A Contribution to the Theory of Economic Growth," Quarterly Journal of Economics 70 (February), pp. 65-94.{Yes}

Tuesday Jan 27: The Solow Growth Model II

David Romer, Advanced Macroeconomics, pp. 15-33.
Robert Solow (1957), "Technical Change and the Aggregate Production Function," Review of Economics and Statistics 39: pp. 312-20. {Yes}

Thursday Jan 29: The Solow Growth Model III

J. Bradford De Long (1988), "Productivity Growth, Convergence, and Welfare: Comment" {Yes}
Martin Feldstein and Charles Horioka (1980), "Domestic Saving and International Capital Flows," Economic Journal 90 (June): pp. 314-329. {Yes}
Xavier Sala-i-Martin (1997), "I Just Ran Four Million Regressions" (NBER working paper 6252). {Yes}