Hamiltonians
(largely cribbed from Maurice Obstfeld's "Guide for the Perplexed"

The Value Function

An alternative way to think about maximizing utility in the Ramsey model--a way that you will see a lot of in the future--involves a particular function called the *Hamiltonian.*

Recall that we had our (intensive form) utility function:

$$U = \int_{t=0}^{\infty} e^{-\beta t} u(c_t)\,dt = \int_{t=0}^{\infty} e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta}\,dt$$

$$\beta = \rho - n - (1-\theta)g$$

And that we have our budget constraint:

$$\lim_{t\to\infty} \{e^{-R_t} e^{(n+g)t} k_t\} \geq 0$$

And that the household's (intensive form) asset holdings evolve according to:

$$\frac{dk_t}{dt} = f(k_t) - c_t - (n+g)k_t$$

Let's set up a *value function:*

$$V[k_0] = \max_{\{c_t\}}[U] = \max_{\{c_t\}} \left[\int_{t=0}^{\infty} e^{-\beta t} u(c_t)\,dt \right]$$

Bellman's Principle

Because it is a value function--adopting *s to denote argument values that maximize the objectives--we can write:

$$V[k_0] = \max_{\{c_t\}} \left[\int_{t=0}^{\infty} e^{-\beta t} u(c_t)\,dt \right] = \int_{t=0}^{T} e^{-\beta t} u(c_t*)\,dt + e^{-\beta T} V[k_T*]$$

This is *Bellman's principle of dynamic programming.*

Now let's consider a discrete time analogue of our problem, with time divided into periods of duration Δh:

$$U = \sum_{j=0}^{\infty} e^{-\beta j \Delta h} u(c_{j\Delta h}) \Delta h$$

subject to:

$$k_{t+\Delta h} - k_t = \Delta h \left[f(k_t) - c_t - (n+g)k_t \right]$$

Then Bellman's principle tells us that:

$$V[k_t] = \max_{c_t}\left\{u(c_t)\Delta h + e^{-\beta\Delta h}V[k_{t+\Delta h}]\right\}$$

And after a little manipulation:

$$0 = \max_{c_t}\left\{u(c_t) - \left[\beta - \frac{\beta^2\Delta h}{2} + \frac{\beta^3(\Delta h)^2}{6} - \ldots\right]V(k_t + \Delta h[f(k_t) - c_t - (n+g)k_t]) + \frac{V(k_t + \Delta h[f(k_t) - c_t - (n+g)k_t]) - V(k_t)}{\Delta h}\right\}$$

The Hamiltonian

Let Δh approach zero, and we have:

$$0 = u(c_t^*) + V'(k_t)[f(k_t) - c_t^* - (n+g)k_t] - \beta V(k_t)$$

$$0 = \max_{c_t}\left\{u(c_t) + V'(k_t)[f(k_t) - c_t - (n+g)k_t]\right\} - \beta V(k_t)$$

The terms inside the braces have a name: the Hamiltonian (William R. Hamilton, b. 1805, Dublin, d. 1865, Dublin). You form the Hamiltonian by taking the contemporanous piece of the function to be maximized--here the utility function--and adding to it the derivative of the value function--the co-state variable--times the rate-of-change of the state variable.

Let me talk a bit about state, control, and co-state variables.

And let me write:

λ = V’(k).

Maximize the term in the braces by choosing consumption c, and find:

$$\frac{\partial u(c_t^*)}{\partial c} = V'(k_t)(-1) = \lambda$$

At each moment the consumer can decide to consume a little bit more, at the price of an infinitesimal drop in the capital stock. An infinitesimal unit of additional consumption yields the marginal payoff of the term on the left, but it also generates an infinitesimal fall in the capital stock--and the derivative of the value function tells you how costly in utility terms this fall in the capital stock is.

This equation has a straightforward economic interpretation. If you increase consumption today, you gain utility. But everything has an opportunity cost. What’s the opportunity cost of increasing consumption today? It’s equal to the effect of changing today’s consumption on future opportunities, times the value of those future opportunities.

Now let’s go back to:

$$0 = u(c_t^*) + V'(k_t)[f(k_t) - c_t^* - (n+g)k_t] - \beta V(k_t)$$

And let’s differentiate the equation with respect to k, taking c* to be a function of k:

$$0 = \left[\frac{\partial u(c_t^*)}{\partial c} + V'(k_t)(-1)\right]\frac{\partial c_t^*}{\partial k_t} + \left[f'(k_t) - (n+g) - \beta\right]V'(k_t) + V''(k_t)[f(k_t) - c_t^* - (n+g)k_t]$$

$$0 = \left[f'(k_t) - (n+g) - \beta\right]V'(k_t) + V''(k_t)[f(k_t) - c_t^* - (n+g)k_t]$$

And set V'(k) = λ:

$$0 = \left[f'(k_t) - (n+g) - \beta \right] \lambda_t + \frac{d\lambda_t}{dk} \frac{dk}{dt}$$

$$\frac{d\lambda_t / dt}{\lambda_t} + \left[f'(k_t) - (n+g) \right] = \beta$$

Now this looks a lot like an asset pricing equation: a "dividend" plus a "capital gain" equals a "required rate of return."

The required rate of return is the (adjusted) rate of time discount beta...

The dividend is the added productive value of having an extra unit of capital on hand: f' - (n+g).

The capital gain is the *rate of change with time* of the variable lambda.

These two conditions--an "asset price" condition that relates the state variable to the derivative with respect to the state variable of the value function, and an "asset accumulation" condition that relates the opportunity cost of changing the state variable to the value of the state variable--are going to come up again and again in intertemporal optimization...

But to return to our problem, simply substitute in consumption for lambda and find that

$$-\theta \frac{dc_t / dt}{c_t} + f'(k_t) - (n+g) = \beta$$

$$\frac{dc_t / dt}{c_t} = \frac{f'(k_t) - (n+g) - \beta}{\theta} = \frac{r_t - \rho - \theta g}{\theta}$$

The Euler equation.

So what's the point of the Hamiltonian methodology? First, it's used a lot in the literature. You should see it now, because you'll see it--in more complexity--later. Second, it's a lot easier to say "write down the Hamiltonian, and solve" then to run through the entire dynamic-programming argument from first principles all the time. Third, the Hamiltonian gives you interesting co-state variables--that have an interesting economic interpretation.

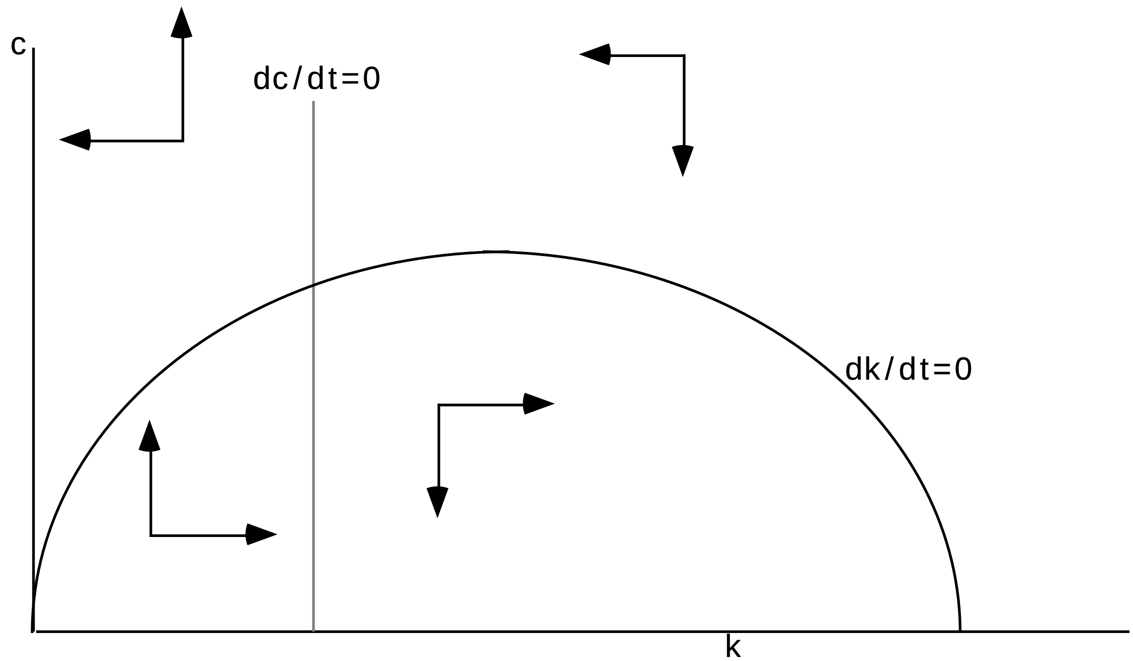
Phase Diagrams

In addition to the equation for consumption growth, we also have our capital accumulation equation.

dk/dt = f(k) - c - (n+g)k

and with the two of them we can start drawing phase diagrams.

Phase Diagrams...



Phase dynamics...

Saddle path...

What good are the other paths? (well suppose you have a terminal condition that you accumulate zero capital as of some point. Look for where you are now, and set c0 so you just hit the vertical axis at that date)...

Why is the dc/dt=0 line to the left of the “Golden Rule” maximum of the dk/dt=0 curve?