Economics 202b: “Bubbles” Handout

In a social-planning or firm intertemporal optimization problem, we often find ourselves faced with a first-order condition for a co-state variable—that is, the shadow value for the planner of having an extra unit of resources in place at time t. This first-order condition often looks something like:

$$P_t = \frac{D_t}{1+r} + E_t\left(\frac{P_{t+1}}{1+r}\right)$$

In a market-equilibrium problem, we often find ourselves with the same first-order condition—this time enforced by market arbitrage:

$$P_t = \frac{D_t}{1+r} + E_t\left(\frac{P_{t+1}}{1+r}\right)$$

And for analytical convenience we often find ourselves assuming (a) that r is constant over time, and (b) that expected dividend growth is at a constant proportional rate g as well:

$$E_t(D_{t+1})=(1+g)D_t$$

In the social-planning context we can solve this problem forward:

$$P_t = \frac{1}{1+r}\left(D_t\left(1+\frac{1+g}{1+r}+\left(\frac{1+g}{1+r}\right)^2+\left(\frac{1+g}{1+r}\right)^3+E_t\left(\frac{P_{t+4}}{(1+r)^3}\right)\right)\right)$$

$$P_t = \lim_{s\to\infty}\left\{\left(\frac{1}{1+r}\right)\sum_{j=0}^{s}\left(\frac{1+g}{1+r}\right)^j D_t + E_t\left(\frac{P_{t+s+1}}{(1+r)^s}\right)\right\}$$

$$P_t = \frac{D_t}{r-g} + \lim_{s\to\infty}\left\{E_t\left(\frac{P_{t+s+1}}{(1+r)^{s+1}}\right)\right\}$$

And we can then use our second-order condition—the assumption that our social planner or our decision-making firm is not really stupid—to conclude that:

$$\lim_{s\to\infty}\left\{E_t\left(\frac{P_{t+s+1}}{(1+r)^{s+1}}\right)\right\} = 0$$

And thus:

$$P_t = P_t^{fund} = \frac{D_t}{r-g}$$

But in the market context, the solution to our arbitrage equation is:

$$P_t = P_t^{fund} + B_t$$

Where Bt is any term that satisfies:

$$E_t(B_{t+1})=(1+r)B_t$$

And what is there in the market to force B_t=0? Appeals to (far distant) irrationality of the bubble when it gets really big? Minimal state variables?

Economics 202b: Froot-Obstfeld Handout

Froot and Obstfeld begin with the arbitrage condition and the dividend process:

$$
\begin{aligned}
&P_t = e^{-r} E_t (D_t + P_{t+1}) \\
&D_t = e^{d_t} \\
&d_{t+1} = \mu + d_t + \varepsilon_{t+1}
\end{aligned}
$$

If we restrict ourselves to solutions that do not depend explicitly on time, the general solution to this is:

$$P_t = \frac{D_t}{e^r - e^{\mu + \frac{\sigma^2}{2}}} + c_1 D_t^{\lambda} + c_2 D_t^{\lambda'}$$

where λ and λ' are the roots of:

$$\lambda^2 \sigma^2 / 2 + \lambda \mu - r = 0$$

To see that this is in fact the solution, substitute back in:

$$
\begin{aligned}
\frac{D_t}{e^r - e^{\mu + \frac{\sigma^2}{2}}} + c_1 D_t^{\lambda} + c_2 D_t^{\lambda'} &= e^{-r} E_t \left\{ D_{t+1} + \frac{D_{t+1}}{e^r - e^{\mu + \frac{\sigma^2}{2}}} + c_1 D_{t+1}^{\lambda} + c_2 D_{t+1}^{\lambda'} \right\} \\
\frac{D_t}{e^r - e^{\mu + \frac{\sigma^2}{2}}} + c_1 D_t^{\lambda} + c_2 D_t^{\lambda'} &= e^{-r} \left\{ D_t + \frac{D_t \left(e^{\mu + \sigma^2/2} \right)}{e^r - e^{\mu + \frac{\sigma^2}{2}}} + c_1 D_t^{\lambda} E_t \left(e^{\mu + \varepsilon_t} \right)^{\lambda} + c_2 D_t^{\lambda'} \left(e^{\mu + \varepsilon_t} \right)^{\lambda'} \right\} \\
\frac{D_t}{e^r - e^{\mu + \frac{\sigma^2}{2}}} + c_1 D_t^{\lambda} + c_2 D_t^{\lambda'} &= \frac{D_t}{e^r - e^{\mu + \frac{\sigma^2}{2}}} + c_1 D_t^{\lambda} e^{-r} \left(e^{\lambda\mu + \lambda^2 \sigma^2/2} \right) + c_2 D_t^{\lambda'} e^{-r} \left(e^{\lambda'\mu + \lambda'^2 \sigma^2/2} \right) \\
\frac{D_t}{e^r - e^{\mu + \frac{\sigma^2}{2}}} + c_1 D_t^{\lambda} + c_2 D_t^{\lambda'} &= \frac{D_t}{e^r - e^{\mu + \frac{\sigma^2}{2}}} + c_1 D_t^{\lambda} + c_2 D_t^{\lambda'}
\end{aligned}
$$

Economics 202b: "Noise" Handout

Milton Friedman has been perhaps the most powerful advocate of the position that asset market institutions—limited job tenure of portfolio managers, short horizons of investors, people who buy and sell for not-very-good reasons—don't matter. The argument that they don't matter has two parts: (a) recursion and (b) selection.

> Recursion—A linked series of investors each in the market for one period should perform the same calculations as one very long-term investor with the same rate of time discount.
>
> Selection—People who don't optimize buy high and sell low: they lose money, and exit from the market.

But recursion only works if there is no "noise" in the market. If there is noise, risk-averse stabilizing speculators will want to take limited positions only. And people who don't optimize lose utility—not necessarily wealth.

In the model investors "live" for two periods, consume in the second period only, and maximize:

> $U = -e^{-(2\gamma)\omega}$, which with normal returns is equivalent to max $E(\omega) - \gamma\sigma^2_{\omega}$

Risky asset—available in unit supply—pays a constant dividend r and sells for a price p_t; safe asset pays a constant dividend r and sells for a price of 1 at all times. "Noise traders" (μ of them) misperceive the expected price of the risky asset next period by a normally distributed (but not mean zero) random variable ρ.

Rational investors (present in the model in measure 1-μ) choose to hold an amount λ^i_t of the risky asset to maximize:

$$E(U) = c_0 + \lambda^i_t\,[r + E_t\,p_{t+1} - p_t(1+r)] - \gamma({}_t\sigma^2_{p(t+1)})(\lambda^i_t)^2$$

Noise traders choose to hold an amount λ^n_t of the risky asset to maximize:

$$E(U) = c_0 + \lambda^n_t\,[r + E_t\,p_{t+1} - p_t(1+r)] - \gamma({}_t\sigma^2_{p(t+1)})(\lambda^n_t)^2 + \lambda^n_t\,\rho_t$$

Agents' holdings of the risky asset are:

$$\lambda^i_t = \frac{r + E_t p_{t+1} - (1+r)p_t}{2\gamma\,({}_t\sigma^2_{p_{t+1}})}$$

$$\lambda^i_t = \frac{r + E_t p_{t+1} - (1+r)p_t + \rho_t}{2\gamma\,({}_t\sigma^2_{p_{t+1}})}$$

Setting total holdings equal to supply gives us:

$$p_t = \frac{r + E_t p_{t+1} - 2\gamma\, {}_t\sigma^2_{p_{t+1}} + \mu\rho_t}{1+r}$$

And looking for a stationary equilibrium gives us:

$$p_t = 1 + \frac{\mu(\rho_t - \rho^*)}{1+r} + \frac{\mu\rho^*}{r} - \frac{2\gamma\mu^2\sigma^2_\rho}{r(1+r)^2}$$

The expected difference in returns between noise traders and sophisticated investors is:

$$E(\Delta R_{n-i}) = \rho^* - \frac{(1+r)^2\rho^{*2} + (1+r)^2\sigma^2_\rho}{2\gamma\mu\sigma^2_\rho}$$

And in an extension of the model in which there is fundamental risk:

$$E(\Delta R_{n-i}) = \rho^* - \frac{(1+r)^2\rho^{*2} + (1+r)^2\sigma^2_\rho}{2\gamma\mu\sigma^2_\rho + \frac{2\gamma(1+r)^2\sigma^2_\varepsilon}{\mu}}$$