1. **(Romer 11.2)** Under $C = Yf(T/Y)$, Romer derives the following condition:

$$f'(T_t/Y_t) = f'(T_{t+1}/Y_{t+1}), \tag{1}$$

which implies that the tax rate is constant over time, ie, $T_t/Y_t = T_{t+1}/Y_{t+1}$, since $f''(\cdot) > 0$.

Under $C = Tg(T/Y)$, the government's optimization problem amounts to:

$$\min_{T_0,T_1,\dots} \sum_{t=1}^{\infty} \frac{1}{(1+r)^t} T_t g(T_t/Y_t) \quad \text{s.t.} \quad \sum_{t=1}^{\infty} \left[\frac{1}{(1+r)^t}(G_t - T_t)\right] + B_0 = 0.$$

Thus, the resulting FOC implies:

$$g(T_t/Y_t) + \frac{T_t}{Y_t} g'(T_t/Y_t) = g(T_{t+1}/Y_{t+1}) + \frac{T_{t+1}}{Y_{t+1}} g'(T_{t+1}/Y_{t+1}). \tag{2}$$

Of course, $T_t/Y_t = T_{t+1}/Y_{t+1}$ is a solution of (2). Furthermore, it is the unique solution. To see this, let consider the following function:

$$F(x, y) = g(x) + xg'(x) - g(y) - yg'(y),$$

where $(x, y) \in R_+^2$. Note that $F(x, x) = 0$ for all x and that

$$\frac{d}{dx}[g(x) + xg'(x)] = g'(x) + xg''(x) + g'(x) > 0,$$

since $x > 0$, $g'(\cdot) > 0$ and $g\text{''}(\cdot) > 0$. Thus, $F(x, x + \varepsilon) < 0$ for all x and all $\varepsilon > 0$, $F(x, x+\varepsilon) > 0$ for all x and all $\varepsilon < 0$, and so $F(x, y) \neq 0$ for all $x \neq y$.

In sum, both specifications of the distortion costs imply that the tax rate must be constant over time.

(a) The fact that the two specifications lead to the same conclusion does not mean that they are equivalent.

Note that $C = Yf(T/Y) = Tg(T/Y)$ if and only if $f(x) = xg(x)$ so that $C = Yf(T/Y) = Y(T/Y)g(T/Y) = Tg(T/Y) = C$.

Let suppose that Romer's specification holds, that is $C = Yf(T/Y)$, $f'(\cdot) > 0$, $f''(\cdot) > 0$ and $f(0) = 0$. Given these assumptions, does Barro's specification necessarily hold as well? Since we need $f(x) = xg(x) \Leftrightarrow g(x) = x^{-1}f(x)$ to hold, and we have some conditions on f,what can we say about g?

- $g'(x) = -x^{-2}f(x) + x^{-1}f'(x)$ whose sign is undetermined since $f(x) > 0$ and $f'(\cdot) > 0$

Hence, Romer's specification does not imply Barro's specification. Or, in other words, Romer's assumptions can hold without Barro's assumptions.

(b) Let suppose that Romer's specification holds, that is $C = Tg(T/Y)$, $g'(\cdot) > 0$, $g''(\cdot) > 0$ and $g(0) = 0$. Given these assumptions, does Romer's specification necessarily hold as well? Since we need $f(x) = xg(x)$ to hold, and we have some conditions on g,what can we say about f?

- $f'(x) = g(x) + xg'(x) > 0$ since $g'(\cdot) > 0$
- $f''(x) = g'(x) + g'(x) + xg''(x) > 0$ since $g'(\cdot), g''(\cdot) > 0$
- $f(0) = 0g(0) = 0$

Hence, Barro's specification implies Romer's specification. Or, in other words, Barro's assumptions are stronger.

(c) Yes. As Romer, Barro assumes that the distortions costs are homogeneous -that is a doubling of tax collections T and potential tax pool Y doubles the collection cost C-, increasing in T, and decreasing in Y. The difference in the two specification is that Barro assumes more convexity in T than what Romer does.